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# Study of General Graph Structure via Eigenvalues

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**Abstract – In this paper, we focus on a particular matrix representation of a graph to understand the structure of graph and using spectral analysis we will devise theorems based on normalized Laplacian matrix in order to prove that if matrixes of two graphs are similar then they are symmetric in spectral plane**.

**Index Terms – Graph, Spectral Graph Theory, Linear Algebra, Matrix, Eigenvalue**

## 1. INTRODUCTION

A graph, denoted by G, is a pair (V, E) of sets such that the elements of E are a collection element subsets of V. We call the elements of V the vertices of the graph, and the elements of E the edges of the graph. We use the notation xy to denote an edge {x, y}, for distinct x, y ∈V. The number of vertices |V | of a graph G is its order.

Spectral graph theory is the study of properties of a graph in relationship to the characteristic polynomial, eigenvalues, and eigenvectors of matrices associated to the graph, such as its adjacency matrix or Laplacian matrix.

Given a graph G, we can form a matrix that contains information about the structure of the graph. Some of the most commonly studied matrix representations of graphs are the adjacency matrix, combinatorial Laplacian, sign-less Laplacian and the normalized Laplacian. The adjacency matrix of a graph G = (V, E ), denoted by A, is a matrix whose rows and columns are indexed by the vertices of G, and is defined to have entries

$$A(x,y) = \left\{ \begin{array}{ll} 1 & \text{if } xy \in E, \\ 0 & \text{otherwise.} \end{array} \right\}$$

The combinatorial Laplacian of a undirected graph G = (V, E ) on n vertices without isolated vertices , denoted by L, is a matrix whose rows and columns are indexed by the vertices of G, and is defined to have entries $dx$ if x = y,

$$L(x, y) = \left\{ \begin{array}{ll} -1 & \text{if } xy \in E, \\ 0 & \text{otherwise.} \end{array} \right\}$$

This matrix is closely related to the adjacency matrix A of G. Let D be a diagonal matrix, whose rows and columns are indexed by the vertices of G, with diagonal entries D(x, x) = $dx$ hence, L = D − A. The matrix D + A is called the sign-less Laplacian of a graph and is denoted by |L|. It should denoted that the sign-less Laplacian is sometimes denoted by $L$+ or Q in the literature, however we will use Q to denote $D^{-1}$A .Finally, the normalized Laplacian of a undirected graph G = (V, E) n vertices without isolated vertices, denoted by £, is a matrix whose rows and columns are indexed by the vertices of G, and is defined to have entries

$$£(x,y) = \left\{ \begin{array}{ll} 1 & \text{if } x=y \text{ and } dy \neq o, \\ -1/\sqrt{dxdy} & \text{if } xy \in E, \\ 0 & \text{otherwise.} \end{array} \right\}$$

In this paper we focus on the spectrum of the normalized laplacian matrix of a graph because Study of the relations between eigenvalues and structures in graphs is the heart of spectral graph theory. We use the superscript notation ($m_i$) to mean that $\lambda_i$ appears in the spectrum with multiplicity $m_i$ and throughout this thesis we assume that graph is simple and there is no isolated vertex.

- Normalized Laplacian
- Eigen Values Of The Normalized Laplacian

### 1.1 Normalized Laplacian

The final type of matrix that we will consider is the normalized Laplacian matrix denoted L. As the name suggests this is closely related to the combinatorial Laplacian that we have just looked at. For graphs with no isolated vertices the relationship is given by

$$L = D^{-1/2}LD^{-1/2} = D^{-1/2}(D - A)D^{-1/2} = I - D^{-1/2}AD^{-1/2}.$$

Throughout the rest of this and ensuing chapters we will usually assume no isolated vertices since they contribute little more than technicalities to the arguments.) Entry wise we have,

$$L_{i,j} = \left\{ \begin{array}{ll} 1 & \text{if } i = j; \\ -1/\sqrt{d_i d_j} & \text{if } i \text{ is adjacent to } j; \\ 0 & \text{otherwise.} \end{array} \right\}$$

For graphs with isolated vertices we let the diagonal entries of that vertex be 0. This gives the nice property that the multiplicity of the eigenvalue 0 is the number of connected components of the graph.

$$\mathcal{L} = \begin{pmatrix} 1 & \frac{-1}{2} & 0 & 0 & 0 \\ \frac{-1}{2} & 1 & \frac{-1}{\sqrt{8}} & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{8}} \\ 0 & \frac{-1}{\sqrt{8}} & 1 & \frac{-1}{\sqrt{6}} & 0 \\ 0 & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{8}} & 0 & \frac{-1}{\sqrt{6}} & 1 \end{pmatrix},$$

with eigenvalues 1.72871355 . . . , 1.5, 1, 0.77128644 . . . , 0. As before we have that 0 is an eigenvalue (now with eigenvector $D^{1/2}1$) and the remaining eigenvalues are nonnegative. A major difference between the two spectra though is that while for the combinatorial Laplacian the eigenvalues can be essentially as large as desired (in particular between 0 and twice the maximum degree), the normalized Laplacian has eigenvalues always lying in the range between 0 and 2 inclusive as shown by Chung. One advantage to this is that it makes it easier to compare the distribution of the eigenvalues for two different graphs, especially if there is a large difference in the "size" of the graphs.

### 1.2 Eigen Values Of The Normalized Laplacian

The normalized Laplacian is a rather new tool which has rather recently (mid 1990's) been popularized by Chung. One of the original motivations for defining the normalized Laplacian was to be able to deal more naturally with non-regular graphs. In some situations the normalized Laplacian is a more natural tool that works better than the adjacency matrix or combinatorial Laplacian. $Q = D^{-1}A$ is the transition matrix of a Markov chain which has the same eigenvalues as I - L.

Let G be a graph of order n. As L is a real symmetric matrix, the eigenvalues are real numbers. We note that normalized Laplacian L of G is a positive semidefinite matrix. To see this, let S be the matrix, whose rows are indexed by the vertices of G and whose columns are indexed by the edges of G (where each edge e = xy is thought of as an ordered 2-tuple e = (x, y), that has entries

$$S(u, e) = \begin{cases} 1/\sqrt{dx} & \text{if } e = xy \text{ and } u = x, \\ -1/\sqrt{dy} & \text{if } e = xy \text{ and } u = y, \\ 0 & \text{otherwise.} \end{cases}$$

The choice of signs can actually be arbitrary so long as in each column (corresponding to an edge of G) there is one positive entry and one negative entry. Then $L = SS^T$ .Therefore all of the eigenvalues of L are nonnegative. Recall that D is the diagonal matrix of vertex degrees of a graph, namely,

$$D(u, v) = \begin{cases} d_u & \text{if } u = v, \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $D^{1/2}1$ is an eigenvector of L with eigenvalue 0. Thus, we assume the eigenvalues of L are

$$0 = \lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda n(L)$$

For graphs without isolated vertices, the normalized Laplacian L has the following relationship to L, A and D

$$L = D^{-1/2}LD^{-1/2};$$

$$= D^{-1/2}(D - A)\, D^{-1/2};$$

$$= I - D^{-1/2}AD^{-1/2}.$$

0 experiment with sudden breaks and gaps in the music. This will give the music more punch and keep people dancing.

## 2. RELATED WORK

In 2003, Haemers et al. conducted a survey of answers to the question of which graphs are determined by the spectrum of some matrix associated to the graph. In particular, the usual adjacency matrix and the Laplacian matrix were addressed. Furthermore, the authors formulated some research questions on the topic. In the meantime, some of these questions have been (partially) answered. In the present paper the authors give a survey of these and other developments.

In 2006, Ivan Gutman and Bo Zhou described about graph in which let *G* be a graph with *n* vertices and *m* edges. Let $\lambda 1$, $\lambda 2$, … , $\lambda n$ be the eigenvalues of the adjacency matrix of *G*, and let $\mu 1$, $\mu 2$, … ,$\mu n$ be the eigenvalues of the Laplacian matrix of *G* . An earlier much studied quantity is the E (G) = $\sum_{n=0}^{n=1} (\lambda_i)$ *energy* of the graph *G*. The authors now define and investigate the *Laplacian energy* as

$$LE(G) = \sum_{n=0}^{N=1} |\mu_i - 2m/n|.$$

There is a great deal of analogy between the properties of *E* (*G*) and *LE* (*G*), but also some significant differences.

In 2010, Cavers M. considered the energy of a simple graph with respect to its normalized Laplacian eigenvalues, which the authors call the L-energy. Over graphs of order n that contain no isolated vertices, the authors characterize the graphs with minimal L-energy of 2 and maximal L-energy of 2bn/2c.

## 3. PORPOSED MODELLING

There are many reasons for this, such as the limitation of finding eigenvalues of general graph. By introducing new theorems we will be able to calculate the eigenvalues of general graph and relevant algorithm for its system implementation. The purpose is to build programs to run on computer system.

## 4. RESULTS AND DISCUSSIONS

As I have discussed the limitation of finding eigenvalues of general graph. To find out Eigen value of general graph I am proposing two theorems with their algorithms and implementation. Following are the theorems:

- Theorem 1
- Theorem 2

### 3.1 Theorem 1

Let G be a graph having vertices $x1$, $x2$, . . . . . . . . . . , $xn$ satisfying that $xi$ is adjacent to $ki$ (>1) number of vertices of degree one, i = 1, 2, . . . . n. then 1 is an Eigen value of (G) With multiplicity at least $\sum_{\mathrm{n}=0}^{\mathrm{N}=1}(K_i - 1)$.

### 3.2 Theorem 2

Let G be a graph having a pair of vertices x and y with deg x = deg y and satisfying the following conditions.

- Both x and y are having m (>1) number of degree one neighbours.
- x and y are having k (≥ 1) no of common neighbours.

Then normalized laplacian £ (G) will have Eigen value

(1 ± √$m$/$m$+$k$)

## 5. CONCLUSION

This paper provides a broad idea about the now research related to find out eigen values of normalized laplacian matrix gives eigenvalue only those graph which follow some pattern like cyclic graph ,path, regular graph, bipartite graph.

## REFERENCES

[1] S. Akbari, E. Ghorbani and M.R. Oboudi, Edge addition, singular values, and energy of graphs and matrices, Linear Algebra Appl., 430 (2009), 2192-2199.
[2] N. Biggs, Algebraic Graph Theory (2nd ed.), Cambridge University Press, Cambridge, 1993.
[3] N. Biggs, E. Lloyd and R. Wilson, Graph Theory, 1736-1936, Oxford University Press,1986.
[4] B. Bollobas and P.Erdos, Graphs of extremal weights, Ars Combin., 50 (1998), 225-233.
[5] O. Bottema, Uber die Irrfahrt in einem Strabennetz. Math. Z., 39 (1935), 137-145.
[6] S.B. Bozkurt, A.D. Gungor, I. Gutman, A.S. Cevik, Randic matrix and Randic energy, MATCH Commun. Math. Computer Chemistry, 64 (2010), 239-250.
[7] R. Brooks, C. Smith, A. Stone and W. Tutte, Dissection of a rectangle into squares, Duke Math. J., 7 (1940), 312-340.
[8] A. Brouwer and W. Haemers, Spectra of Graphs, unpublished course notes, retrieved Feb 1st, 2010, (http://homepages.cwi.nl/~ aeb/math/ipm.pdf).
[9] S. Butler, A note about cospectral graphs for the adjacency and normalized Laplacian matrices, Linear Multilinear Algebra, 58(3) (2010), 387-390.
[10] S. Butler, Eigenvalues and Structures of Graphs, Ph.D. dissertation, University of California, San Diego, 2008.
[11] M. Cavers, S. Fallat and S. Kirkland, on the normalized Laplacian energy and general Randic index $R_{-1}$ of graphs, Linear Algebra Appl., 433(1) (2010), 172190.
[12] F.R.K. Chung, Spectral Graph Theory, American Math. Soc., Providence, 1997.
[13] F. Chung and R. Langlands, A combinatorial Laplacian with vertex weights, J. Combin. Theory Ser. A, 75 (1996), 316-327.
[14] L.H. Clark and J.W. Moon, on the general Randic index for certain families of trees, Ars Combin., 54 (2000), 223-235.
[15] L. Collatz and U. Sinogowitz, Spektren endlicher Grafen, Abh. Math. Sem. Univ. Hamburg, 21 (1957), 63-77.